



ELSEVIER

Available online at www.sciencedirect.com

SCIENCE @ DIRECT®

Journal of
Combinatorial
Theory
Series B

Journal of Combinatorial Theory, Series B 92 (2004) 177–181

<http://www.elsevier.com/locate/jctb>

Note

The circular chromatic number of induced subgraphs

Xuding Zhu¹

*Department of Applied Mathematics, National Sun Yat-sen University, 70 Lien-hai Road,
Kaohsiung 80424, Taiwan*

Received 19 November 2002

Available online 28 May 2004

Abstract

This note presents an infinite family of graphs G for which $\chi_c(G) = 4$ and for each vertex x of G , $\chi_c(G - x) = 8/3$. This gives a negative answer to a question asked in Zhu (Discrete Math. 229 (1–3) (2001) 371).

© 2004 Elsevier Inc. All rights reserved.

Keywords: Circular chromatic number; Subgraph; Orientation

Suppose $G = (V, E)$ is a graph and $k \geq 2d$ are positive integers. A (k, d) -coloring of G is a mapping $f: V \rightarrow \{0, 1, \dots, k-1\}$ such that for any edge xy of G , $d \leq |f(x) - f(y)| \leq k - d$. The *circular chromatic number* $\chi_c(G)$ of a graph G is defined as $\chi_c(G) = \min\{k/d: \text{there exists a } (k, d)\text{-coloring of } G\}$. It is known [1,5,8] that $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$, so $\chi_c(G)$ is a refinement of $\chi(G)$.

As $\chi(G) - 1 \leq \chi(G - x) \leq \chi(G)$ for every vertex x of G , it follows that $\chi_c(G) - 2 < \chi_c(G - x) \leq \chi_c(G)$. How small can $\chi_c(G - x)$ be? It was shown in [7] that for any $\varepsilon > 0$ there is a graph G which contains a vertex x such that $\chi_c(G - x) < \chi_c(G) - 2 + \varepsilon$. But it is natural to ask whether every graph G must contain a vertex x with $\chi_c(G - x) \geq \chi_c(G) - 1$. This question was posed in [7] and remained open until now: in this paper we give a negative answer by constructing an infinite family of graphs G with $\chi_c(G) = 4$ and $\chi_c(G - x) = 8/3$ for every vertex x of G .

A corresponding question can be asked regarding edge deletion. In this case it was proved already in [7] that $\chi_c(G - e) \geq \chi_c(G) - 1$ for every edge e of G . Recently,

E-mail address: zhu@math.nsysu.edu.tw.

¹Supported in part by the National Science Council under Grant NSC89-2115-M-110-012.

Hajiabolhassan and Zhu [4] strengthened this inequality to $\chi_c(G - e) \geq \chi(G) - 1$. From this result it follows that any graph G with $\chi_c(G) > n$ for an integer n has a subgraph H with $\chi_c(H) = n$. We cannot always take H to be an induced subgraph here, since this would require that any graph G has a vertex x for which $\chi_c(G - x) \geq \chi(G) - 1$, and by our construction this might not be possible.

Let C_{2k+1} be the $(2k + 1)$ -cycle and P_n be the path with n edges. The Cartesian product $P_n \square C_{2k+1}$ has vertices $x_{i,j}$: $i = 0, 1, \dots, n, j = 0, 1, \dots, 2k$, and in which $x_{i,j} \sim x_{i',j'}$ if either $j = j'$ and $i = i' \pm 1$, or $i = i'$ and $j = j' \pm 1 \pmod{2k+1}$.

Let $G(n, k)$ be the graph obtained from $P_n \square C_{2k+1}$ by identifying $x_{0,j}$ with $x_{n,2k+1-j}$ for $j = 0, 1, \dots, 2k$. (Note that the calculation in the second coordinate of the index is modulo $2k + 1$. Thus $2k + 1 - 0 = 0$, i.e., $x_{0,0}$ is identified with $x_{n,0}$.)

Theorem 1. *For any $k \geq 1$ and $n \geq 2$, $\chi_c(G(n, k)) = 4$. If $k \geq 2$ and $n \geq 6$, then for any vertex x of $G(n, k)$, $\chi_c(G(n, k) - x) = 8/3$.*

Proof. The graph $G = G(n, k)$ is actually a quadrangulation of the Klein bottle, and it follows from a result in [2] that $\chi_c(G) = 4$. We also note that the special case $k = 1$ of the first part of the theorem was proved by Zhou [6], where the proof is quite long. For the completeness of this note, we give a direct proof, as the argument is also needed to prove that $\chi_c(G - x) = 8/3$ for any vertex x of G .

As G is 4-regular, it follows from Brooks' theorem that G is 4-colorable. Thus $\chi_c(G) \leq 4$. Assume to the contrary that $\chi_c(G) = r < 4$. By a result in [3], G has an orientation D such that for any circuit C of G , $|C|/|C^+(D)| \leq r$ and $|C|/|C^-(D)| \leq r$.

Such an orientation D of G induces an orientation D' of $P_n \square C_{2k+1}$, in which the circuit $C(0) = (x_{0,0}, x_{0,1}, x_{0,2}, \dots, x_{0,2k})$ and the circuit $C(n) = (x_{n,0}, x_{n,2k}, x_{n,2k-1}, \dots, x_{n,1})$ have the same orientation (because they are a single circuit in G). Thus $|C(0)^+(D)| = |C(n)^+(D)|$ and $|C(0)^-(D)| = |C(n)^-(D)|$.

For $i = 0, 1, \dots, n-1$ and $j = 0, 1, \dots, 2k$, let $C(i, j)$ be the circuit $C(i, j) = (x_{i,j}, x_{i,j+1}, x_{i+1,j+1}, x_{i+1,j})$. (The addition in the second coordinate of the indices is taken modulo $2k + 1$.) Since $|C|/|C(i, j)^+(D)| < 4$ and $|C|/|C(i, j)^-(D)| < 4$, it follows that $|C(i, j)^+(D)| = |C(i, j)^-(D)| = 2$. Therefore

$$\sum_{i=0}^{n-1} \sum_{j=0}^{2k} (|C(i, j)^+(D)| - |C(i, j)^-(D)|) = 0.$$

In the summation above, each edge of $P_n \square C_{2k+1}$ other than those edges in $C(0)$ and in $C(n)$ is contained in two of the circuits $C(i, j)$, once as a forward edge, and once as a backward edge. Therefore each of these edges contributes 0 to the summation. The edges in $C(0)$ and $C(n)$ are counted once, in the direction of $C(0)$ and $C(n)$, respectively. So

$$\begin{aligned} 0 &= \sum_{i=0}^{n-1} \sum_{j=0}^{2k} (|C(i, j)^+(D)| - |C(i, j)^-(D)|) \\ &= (|C(0)^+(D)| - |C(0)^-(D)|) + (|C(n)^+(D)| - |C(n)^-(D)|) \\ &= 2(|C(0)^+(D)| - |C(0)^-(D)|). \end{aligned}$$

Hence $|C(0)^+(D)| = |C(0)^-(D)|$. But this is impossible, as $|C(0)^+(D)| + |C(0)^-(D)| = 2k + 1$ is odd, and each of $|C(0)^+(D)|, |C(0)^-(D)|$ is an integer.

Next we show that for any vertex x of G , $\chi_c(G - x) = 8/3$. For $i \geq 1$, the mapping ϕ defined as

$$\phi(x_{i'j}) = \begin{cases} x_{i'-i,j}, & \text{if } i \leq i' \leq n-1 \\ x_{i'-i+n,2k+1-j}, & \text{otherwise} \end{cases}$$

is an automorphism of $G(n, k)$. Also the mapping ψ defined as $\psi(x_{ij}) = x_{i,2k+1-j}$ is an automorphism of $G(n, k)$. Therefore, we may assume the deleted vertex $x = x_{1,j^*}$ for some $0 \leq j^* \leq k$.

First we show that $\chi_c(G - x) \geq 8/3$. Assume $\chi_c(G - x) = r < 8/3$. Again let D be an orientation of $G - x$ for which $|C|/|C^+(D)| \leq r$ and $|C|/|C^-(D)| \leq r$ for each circuit C of G . Similarly let D' be the orientation of $P_n \square C_{2k+1} - x_{1,j^*}$ induced by D . We consider the circuits C_{ij} as defined in the proof above, except that the four circuits $C_{0,j^*-1}, C_{0,j^*}, C_{1,j^*-1}, C_{1,j^*}$ are replaced by a single circuit C^* which is the symmetric difference of $C_{0,j^*-1}, C_{0,j^*}, C_{1,j^*-1}, C_{1,j^*}$. Now $|C^*| = 8$. Since $|C^*|/|(C^*)^+(D)| < 8/3$ and $|C^*|/|(C^*)^-(D)| < 8/3$, it follows that $|(C^*)^+(D)| = |(C^*)^-(D)| = 4$. The rest of the argument is the same as in the previous paragraph, which leads to the same contradiction.

It remains to prove that $\chi_c(G - x) \leq 8/3$. If $n \geq 8$, then the mapping f defined as

$$f(x_{i,j}) = \begin{cases} x_{i,j}, & \text{if } i \leq n-3 \\ x_{i-2,j}, & \text{if } i \geq n-2 \end{cases}$$

is a homomorphism from $G(n, k) - x$ to $G(n-2, k) - x$. Therefore, it suffices to consider the case that $n = 6$ and 7 . If $k \geq 3$ and $j^* \neq k-1, k, k+1, k+2$, then let f be the mapping defined as

$$f(x_{i,j}) = \begin{cases} x_{i,j}, & \text{if } 0 \leq j \leq k, \\ x_{i,j-2}, & \text{if } k+1 \leq j \leq 2k. \end{cases}$$

It is straightforward to verify that f is a homomorphism from $G(n, k) - x_{1,j^*}$ to $G(n, k-1) - f(x_{1,j^*})$. If $k \geq 5$ and $j^* \in \{k-1, k, k+1, k+2\}$, then let f be the mapping defined as

$$f(x_{i,j}) = \begin{cases} x_{i,j}, & \text{if } j = 0, 1, \\ x_{i,j-2}, & \text{if } 2 \leq j \leq 2k-2, \\ x_{i,0}, & \text{if } j = 2k-1, \\ x_{i,2k-4}, & \text{if } j = 2k. \end{cases}$$

Then f is a homomorphism from $G(n, k) - x_{1,j^*}$ to $G(n, k-2) - f(x_{1,j^*})$. Therefore it suffices to show the following:

1. $G(6, 2) - x_{1,j^*}$ is $(8, 3)$ -colorable for $j^* = 0, 1, 2$.
2. $G(7, 2) - x_{1,j^*}$ is $(8, 3)$ -colorable for $j^* = 0, 1, 2$.
3. $G_{6,3} - x_{1,j^*}$ is $(8, 3)$ -colorable for $j^* = 2, 3$.
4. $G_{7,3} - x_{1,j^*}$ is $(8, 3)$ -colorable for $j^* = 2, 3$.

5. $G_{6,4} - x_{1,j^*}$ is $(8, 3)$ -colorable for $j^* = 3$.
6. $G_{7,4} - x_{1,j^*}$ is $(8, 3)$ -colorable for $j^* = 3$.

Figs. 1–3 give the required $(8, 3)$ -colorings of $G - x$ for each of the above cases. \square

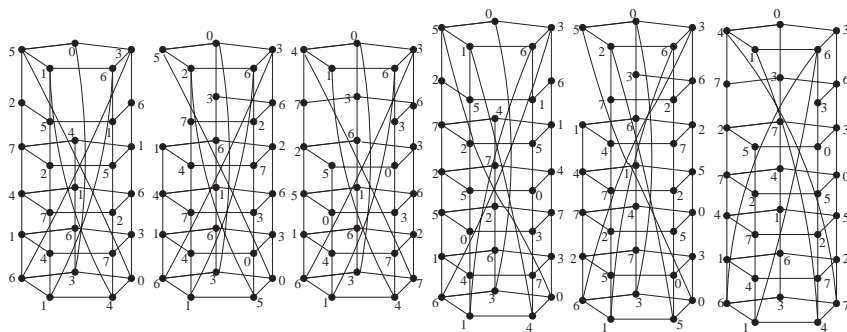


Fig. 1. $(8, 3)$ -coloring of $G(6, 2) - x$ and $G(7, 2) - x$.

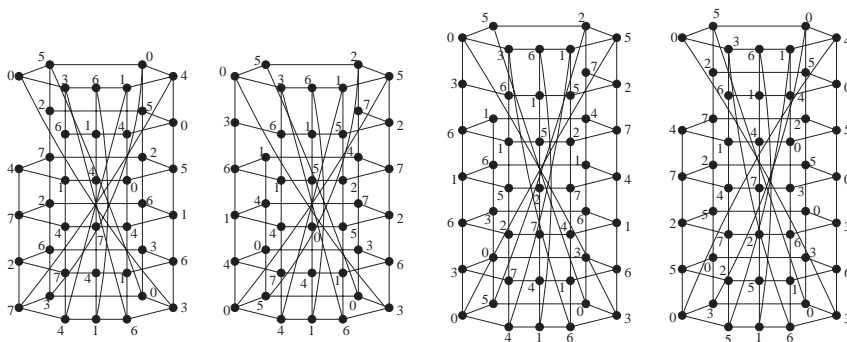


Fig. 2. $(8, 3)$ -coloring of $G(6, 3) - x$ and $G(7, 3) - x$.

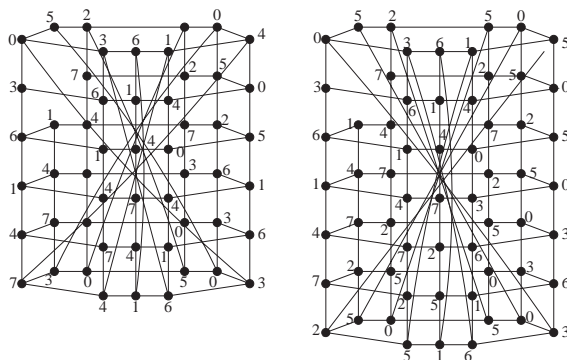


Fig. 3. $(8, 3)$ -coloring of $G(6, 4) - x$ and $G(7, 4) - x$.

Although we have an infinite family of graphs G such that $\chi_c(G - x) < \chi_c(G) - 1$ for each vertex x , all these graphs G have the same circular chromatic number, and all the subgraphs $G - x$ also have the same circular chromatic number. It would be interesting to know whether these are the exception cases, or there are other examples. In particular, it would be interesting to answer the following questions.

Question 1. *Are there n -chromatic graphs G for some $n \geq 5$ such that for any vertex x , $\chi_c(G) - \chi_c(G - x) > 1$?*

Question 2. *Are there graphs G such that for every vertex x of G , $\chi_c(G) - \chi_c(G - x) \geq 2 - \varepsilon$ for some $\varepsilon < 2/3$? Or even for any $\varepsilon > 0$?*

Question 3. *Are there graphs G for which $\chi_c(G) \neq \chi(G)$ and yet there is a vertex x of G such that $\chi_c(G) - \chi_c(G - x) > 1$?*

References

- [1] J.A. Bondy, P. Hell, A note on the star chromatic number, J. Graph Theory 14 (1990) 479–482.
- [2] M. DeVos, L. Goddyn, B. Mohar, D. Vertigan, X. Zhu, Coloring-flow duality of embedded graphs, Trans. Amer. Math. Soc., to appear.
- [3] L.A. Goddyn, M. Tarsi, C.Q. Zhang, On (k, d) -colorings and fractional nowhere-zero flows, J. Graph Theory 28 (1998) 155–161.
- [4] H. Hajiabolhassan, X. Zhu, The circular chromatic number of subgraphs, J. Graph Theory 44 (2003) 95–105.
- [5] A. Vince, Star chromatic number, J. Graph Theory 12 (1988) 551–559.
- [6] B. Zhou, Some theorems concerning the star chromatic number of a graph, J. Combin. Theory (B) 70 (1997) 245–258.
- [7] X. Zhu, Star chromatic numbers and products of graphs, J. Graph Theory 16 (1992) 557–569.
- [8] X. Zhu, Circular chromatic number, a survey, Discrete Math. 229 (1–3) (2001) 371–410.